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ScienceDirect
Procedia Computer Science 4 (2011) 1383–1392
Procedia Computer Science

International Conference on Computational Science, ICCS 2011

Algorithm for waiting time distribution of a discrete-time multiserver queue with deterministic service times and multi-threshold service policy

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Abstract

In this paper, a discrete-time multiserver queueing system with infinite buffer size is studied. Packets arrive at the system according to a Bernoulli arrival process, and the service times of the packets are assumed to be constant equal to an integer multiple of slots. The number of active servers in the system is controlled by a so-called multi-threshold service policy with a set of thresholds predefined. For this multiserver system, we derive its stationary waiting time distribution by using the stochastic complement and Crommelin's techniques. We also present an algorithm for computing the stationary waiting time distribution.

Keywords: discrete-time queue, multi-threshold service policy, stochastic complement, Crommelin's technique

1. Introduction

Discrete-time queueing models have been used to analyze the performance of digital communication systems, such as multiplexers and packet switches. In such models, the time axis is divided into fixed-length time intervals, referred to as slots. Arrivals and services (transmission) of packets can start or end at slot boundaries only. The models have been studied under various assumptions with respect to packet arrival processes and service-time distributions (see [2]-[7], [10]-[12], [16]). In particular, the works for the discrete-time multiserver systems with constant service times have received extensively attention in the literature. Crommelin [7] presented a very simple approach for the $M/D/c$ system by studying it as a batch service system. Crommelin's idea was also discussed and generalized to the $MAP/D/c$ system in Neuts [15]. Now, the method is known as Crommelin's technique and has been applied by other researchers in analyzing the multiserver systems with constant service times. Nishmura [16] studied the $MAP/D/N$ system in continuous-time based on this idea and used spectral technique for the analysis. Franx [10] presented a simple solution for the $M/D/c$ waiting times. Recently, Chaudhry et al. [6] and Alfa [3] analyzed the discrete-time

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Selection and/or peer-review under responsibility of Prof. Mitsuhisa Sato and Prof. Satoshi Matsuoka
doi:10.1016/j.procs.2011.04.149

$MAP/D/k$ system using Crommelin's technique and presented the efficient computational algorithms for the waiting time distribution.

In this paper, we also consider a discrete-time multiserver system with constant service times, but the system is controlled by a so-called multi-threshold service policy. Under this service discipline, the number of servers employed in service at any discrete-time is governed by a threshold sequence $F \equiv \{F_1, F_2, \cdots, F_{K-1}\}$ of positive integers such as $F_1 < F_2 < \cdots < F_{K-1}$. For notational convenience, we set $F_0 = 0$ and $F_K = \infty$. When a customer arrives at an empty system, he/she is served by a single server. Whenever the number of customers is greater than or equal to a threshold F_k, a server is added to the system and immediately starts to serve. Similarly, whenever the number of customers falls below the threshold F_k, a server is removed from the system. As a simple and efficient control scheme, the threshold-based service policies have been widely used in the continuous-time multiserver queueing systems (see [12], [13], [8], [9]). To the best of our knowledge, however, no researches have been conducted for the discrete-time system with the multi-threshold service policy. For such the system, the variety of the number of active servers (being in service) complicates the analysis of the system performance. First, Crommelin's idea can not be applied directly as it has been done in [3], [6], [7] and [16], where the analysis was carried out by focusing on the behavior of a fixed server. Secondly, the waiting time of any customer is affected not only by the number of the customers ahead of him on arrival epoch, but also by the further arriving situation of customers, since this may result in an increase or decrease of the number of active servers. For this sake, in this paper we derive the stationary waiting time distribution in two steps: first we verify that the waiting time distribution in our queueing system is the same as that of the last arriving customer in a discrete-time single server queueing system with bulk service, the size of which is controlled by F. For this single server queueing system, we derive the steady state distribution of the queue length using a combination technique of stochastic complement and Crommelin's methods. Next, we compute the stationary waiting time distribution by introducing an infinite absorbing discrete-time Markov chain with an appropriate state space. The waiting time of a customer is given by the time until absorption. Hence, the stationary waiting time distribution is of phase type.

The remainder of the paper is organized as follows. In Section 2, we describe the queueing model in detail and give some preliminary results. In Section 3, we perform the analysis of the queueing system. Instead of working on the original multiserver queueing system, we consider a single server queue that has the same waiting time distribution and derive its steady-state distribution of the queue length using Crommelin's and stochastic complement ideas. In Section 4, we introduce an absorbing Markov chain to describe the dynamics of the system after the tagged customer arrived and derive the stationary waiting time distribution. We also present an algorithm for computing this stationary distribution. Finally, some conclusions are included in Section 5.

2. Model and Preliminaries

In this section, we describe the discrete-time queueing model in detail, and present a brief introduction to the concepts of Crommelin and stochastic complementation, and some related results.

2.1 The discrete-time queueing model

Consider a $Geo/D/K$ queueing system in discrete-time. Customers arrive at the system according to a Bernoulli process with parameter λ. The service times of the customers are constant equal to d slots. The distribution of the deterministic service times can be represented as the phase type distribution $(\boldsymbol{\alpha}, S)$ as in [3] and [6], where $\boldsymbol{\alpha} = [1, 0, \cdots, 0]$, $S = \begin{bmatrix} \mathbf{0} & I_{d-1} \\ 0 & \mathbf{0} \end{bmatrix}$ and I_j is an identify matrix of order j. All arrivals occur just after the beginning of a slot, and departures take place just before the end of a slot. K servers serve the customers by following the multi-threshold service scheme with F as follows. When a customer arrives at an empty system, it is served by a single server. Whenever the number of customers is greater than or equal to the threshold F_k, a server is added to the system and immediately starts to serve. Similarly, whenever the number of customers falls below the threshold F_k, a server is removed from the system. We define the traffic intensity $\rho = \lambda d/K$ and assume that $\rho < 1$ for the stability of the system. Furthermore, we assume that the service discipline is FIFO and that the servers serve customers in the cyclic service order.

2.2 Crommelin's method

For an $M/D/k$ queueing system, Crommelin's method presents actually a study of customers $j, k+j, 2k+j, \cdots, vk+j, \cdots$ (e.g., $j = 1$) in a single server system where the single server is the jth server in the multiserver system, under assumptions of the FIFO service discipline and cyclic service order. Alternatively, this can also be looked as studying a system of group service queue with a single server in which the server attends to exactly a group of k customers at a time [3]. Crommelin [7] showed that the stationary distribution of the queue length at arbitrary time in the $M/D/k$ model is the invariant probability vector in this group service queue. Neuts [15] extended the Crommelin's results to the $MAP/D/k$ model. Recently, this idea has also been used in studying the waiting time distribution for a discrete-time $MAP/D/k$ model by Chaudhry et al. [6] and Alfa [3], based on the fact that the waiting time distribution of an arbitrary customer in the $MAP/D/k$ model is the same as that of the last arriving customer in a group of k customers at a time.

For the discrete-time $Geo/D/K$ queueing model considered in the present paper, since the number of servers added into service is controlled by the multi-threshold service scheme, the associated single server model with group service, in fact, is one with varying group service size. Concretely, when the number of the customers is between the thresholds F_{k-1} and F_k, the size of group service in the single server model is k. Hence, Crommelin's approach that only observes the behavior of a fixed server is not directly applicable in our model. For this reason, we will decompose the system into K sub-systems using the concept of stochastic complementation described below, and then for each sub-system, we apply Crommelin's idea to derive its steady-state distribution.

2.3 Stochastic complementation

Here, we state briefly the concept of stochastic complementation and quote some useful results from [12]. For a given irreducible discrete time Markov chain $\mathcal{M}$ with state space $\mathcal{S}$, let us partition this state space into two disjoint sets A and B. Then, the one-step transition probability matrix of $\mathcal{M}$ is

$$\boldsymbol{P} = \begin{pmatrix} P_{A,A} & P_{A,B} \\ P_{B,A} & P_{B,B} \end{pmatrix}$$

and the corresponding steady state probability vector is $\pi = [\pi_A, \pi_B]$. The stochastic complement of $P_{A,A}$ is defined by the matrix $C_{A,A} = P_{A,A} + P_{A,B}[I - P_{B,B}]^{-1}P_{B,A}$. For the stochastic complement matrix $C_{A,A}$ and steady state probability π_A, then, the following results hold.

Proposition 2.1 (i) $C_{A,A}$ is always a stochastic matrix and the associated Markov chain is always irreducible, if the original Markov chain is irreducible. (ii) Let $\pi_{|A}$ be the steady state probability vector for the stochastic complement $C_{A,A}$, then $\pi_{|A} = 1/(\pi_A e)\pi_A$, where e is the column vector with all entries equal to 1.

The implication of the above theorem is that the steady state probabilities of the stochastic complement are the *conditioned steady state probabilities* of the associated states of the original Markov chain. For more details on the stochastic complement concept, refer to [12] and the references therein.

3. Analysis of the model

We observe the $Geo/D/K$ queueing system with the multi-threshold service scheme just after the arrivals. Then, its dynamics can be described by a multi-dimensional Markov process $\mathcal{M} = \{(L_n, S_n, K_n),\ n \geq 0\}$, where L_n, $S_n = (S_{1n}, S_{2n}, \cdots, S_{K_n n})$ and K_n denote, respectively, the number of customers, the remaining service time vector and number of the active servers (being in service) at the nth slot. The state space of $\mathcal{M}$ is given by $\mathcal{S} = \{(l, (s_1, s_2, \cdots, s_k), k) \mid l \geq 0,\ 1 \leq s_i \leq d; 1 \leq i \leq k,\ 1 \leq k \leq K\}$. Note that the random variable K_n may be dropped, because according to the multi-threshold service scheme, we know that the number of the active servers is k when the number of customers is in the range $F_{k-1} \leq l < F_k$. However, we remain it here for the sake of descriptive completeness. By the analogous arguments in Crommelin [7], Neuts [14] and Chaudhry et al. [6], we conclude that under the assumptions of FIFO and cyclic service scheme, the waiting time distribution of an arbitrary customer in the system with the multi-threshold service scheme is the same as that of the last arriving customer in a group in a single $MAP/D/1$ queueing system with bulk service, the size of which is controlled by the multi-thresholds F. Instead of our original $Geo/D/K$ queueing system, therefore, we analyze its associated single server $MAP/D/1$ queueing system. The dynamics of this queueing system can be described by the multi-dimensional Markov process $\mathcal{M}^* = \{(L_n, J_n, S_n, K_n),\ n \geq 0\}$, where L_n, J_n, S_n and K_n denote, respectively, the number of customers, phase of the Markov arrival process, remaining group service time and the bulk size (i.e., the number of the active servers) at the nth slot. We refer $\mathcal{M}^*$ to as the associated process of the original Markov chain $\mathcal{M}$. The state space of $\mathcal{M}^*$ is given by $\mathcal{S}^* = \{(l, j, s, k), \mid l \geq 0,\ 1 \leq j \leq k,\ 1 \leq s \leq d,\ 1 \leq k \leq K\}$. As mentioned before, due to the controlled number of active servers, Crommelin's idea can not be applied directly in deriving the steady state distribution of $\mathcal{M}^*$. In this section, we divide the derivation of the stationary distribution into the following three steps. First, according to the number of the active servers, we partition the state space into K disjoint state sub-spaces using the concept of stochastic complementation, and corresponding to each sub-state space, we define an Markov chain on it. Since the number of active servers is fixed on each sub-state space, this allows us to apply Crommelin's idea in computing the steady-state distribution of the sub-Markov chain. In fact, this steady state distribution is a conditional steady state probability vector, given that the Markov chain $\mathcal{M}^*$ is in that sub-state set. Secondly, we aggregate each sub-state space into a single state and compute the steady state probabilities of the system being in any aggregated state. Lastly, we utilize these steady state distributions to compute the (unconditional) steady state probabilities of the associated Markov chain $\mathcal{M}^*$.

3.1 The partition of the state space $\mathcal{S}$ and definition of the associated Markov processes

For the sake of clarity, we begin with the decomposition of the original Markov process $\mathcal{M}$, and then establish a decomposition of the associated Markov process $\mathcal{M}^*$. For the Markov process $\mathcal{M}$, let us partition its state space $\mathcal{S}$ into K disjoint sets $\mathcal{S}_k$ such as: $\mathcal{S}_1 = \{(0,0,0), (l,(s_1),1) \mid 1 \leq l < F_1,\ 1 \leq s_1 \leq d\}$, $\mathcal{S}_k = \{(l,(s_1,\cdots,s_k),k) \mid F_{k-1} \leq l < F_k,\ 1 \leq s_i \leq d;\ 1 \leq i \leq k\}$, $k = 2,\cdots,K-1$ and $\mathcal{S}_K = \{(l,(s_1,\cdots,s_K),K) \mid F_{K-1} \leq l < \infty,\ 1 \leq s_i \leq d;\ 1 \leq i \leq K\}$. In each sub-state space $\mathcal{S}_k$, the lexicographical order is defined. Corresponding to $\mathcal{S}_k$, we introduce an Markov chain $\mathcal{M}_k$ with the transition structure similar to that of $\mathcal{M}$ in $\mathcal{S}_k$, except for the following modifications: (1) for $k = 1$ and $2 \leq s_1 \leq d$, a transition from $(F_1 - 1, (s_1), 1)$ to $(F_1, (s_1 - 1), 2)$ with probability λ in the original process $\mathcal{M}$ is replaced by a transition from $(F_1 - 1, (s_1), 1)$ to $(F_1 - 1, (s_1 - 1), 1)$ with probability 1. (2) for $2 \leq k \leq K-1$, note that due to a single arrival, the cyclic service order and constant service times, we have that all the remaining service times, $s_1, \cdots, s_k$ are different. Without loss of generality, we assume that $s_1 > s_2 > \cdots > s_k$. Then, a transition from $(F_{k-1}, (s_1,\cdots,s_{k-1},1),k)$ to $(F_{k-1}-1,(s_1-1,\cdots,s_{k-1}-1),k-1)$ with probability $1-\lambda$ in the Markov chain $\mathcal{M}$ is replaced by a transition from $(F_{k-1},(s_1,\cdots,s_{k-1},1),k)$ to $(F_{k-1},(s_1-1,\cdots,s_{k-1}-1,d),k)$ with probability 1, and for $2 \leq s_i \leq d$, $i = 1,\cdots,k$, a transition from $(F_k - 1,(s_1,\cdots,s_k),k)$ to $(F_k,(s_1-1,\cdots,s_k-1,d),k+1)$ with probability λ in the Markov chain $\mathcal{M}$ is replaced by a transition from $(F_k-1,(s_1,\cdots,s_k),k)$ to $(F_k-1,(s_1-1,\cdots,s_k-1),k)$ with probability 1, and (3) for $k = K$, a transition from $(F_{K-1},(s_1,\cdots,s_{K-1},1),K)$ to $(F_{K-1}-1,(s_1,\cdots,s_{K-1}),K-1)$ with probability $1-\lambda$ in the Markov chain $\mathcal{M}$ is replaced by a transition from $(F_{K-1},(s_1,\cdots,s_{K-1},1),K)$ to $(F_{K-1},(s_1,\cdots,s_{K-1},d),K)$ with probability 1. These modifications are mainly based on the observation that there exists only a single return and entry among $\mathcal{S}_k$, $k = 1,2,\cdots,K$ in the original Markov chain, and the transition rates do not depend on the states of the remaining service time. According to this fact and Proposition 2.1, we obtain the following result for the Markov chains $\mathcal{M}_k$.

Proposition 3.1. The steady state probabilities of the Markov chain $\mathcal{M}_k$ is the conditional steady state probabilities for the states in $\mathcal{S}_k$ of the original Markov chain $\mathcal{M}$, given that the system is in partition $\mathcal{S}_k$.

For each k, $\mathcal{M}_k$ describes the dynamics of the $Geo/D/k$ queueing system with the finite state space $\mathcal{S}_k$. For this model, we can use Crommelin's idea because the number of active servers is k, fixed. That is, we can consider an equivalent queueing system by observing the behavior of the server who is just added in service when the number of the customers exceeds F_{k-1}. This queueing model is one with a single server and Markovian arrival process, namely, $D - MAP(C^{*(k)}, D^{*(k)})/D/1$. For the proof of the equivalence, see Proposition 1 in [6]. Here, $C^{*(k)}$ and $D^{*(k)}$ are, respectively, $k-$dimensional square matrices implying the transition probability of the phase of the underlying Markov chain without and with arrivals such that

$$\boldsymbol{C^{*(k)}} = \begin{pmatrix} \bar{\lambda} & \lambda & & \\ & \ddots & \ddots & \\ & & \ddots & \lambda \\ & & & \bar{\lambda} \end{pmatrix}, \qquad \boldsymbol{D^{*(k)}} = \begin{pmatrix} & & & \\ & & & \\ & & & \\ \lambda & & & \end{pmatrix}$$

where $\bar{\lambda} = 1 - \lambda$. For the $D - MAP(C^{*(k)}, D^{*(k)})/D/1$ queueing system, we consider the Markov process $\mathcal{M}_k^* = \{(L_n, J_n, S_n, k); n \geq 0\}$, where L_n, J_n and S_n denote, respectively, the number of customers, phase of the Markov arrival process $MAP(C^{*(k)}, D^{*(k)})$, and remaining service time at the nth slot. Then, $\mathcal{M}_k^*$ is a finite quasi-birth-death (QBD) process on the state space $\mathcal{S}_k^* = \{(l, j, s, k) \mid F_{k-1} \le l < F_k;\ 1 \le j \le k;\ 1 \le s \le d\}$. Its transition probability matrix will be given below. The processes $\mathcal{M}_k^* = \{(L_n, J_n, S_n, k);\ n \ge 0\}$, in fact, describe the dynamics of $\mathcal{M}^*$ when its state is limited in the state space $\mathcal{S}_k^*$. Therefore, we have that the steady state probabilities of $\mathcal{M}_k^*$ is the conditional steady state probabilities for the states in $\mathcal{S}_k^*$ of the associated Markov chain $\mathcal{M}^*$, given that the system is in partition $\mathcal{S}_k^*$.

3.2 The steady state probabilities of $\mathcal{M}_k^*$

In this sub-section, we derive the steady state probabilities of the Markov chain $\mathcal{M}_k^*$, $k = 1, 2, \cdots, K$. For $\mathcal{M}_1^*$, we have $C^{*(1)} = (\bar{\lambda})_{1\times 1},\ D^{*(1)} = (\lambda)_{1\times 1}$ because of only one active server (hence, $j \equiv 1$, we can think $\mathcal{M}_1 = \mathcal{M}_1^*$). The transition probability matrix of the Markov chain $\mathcal{M}_1^*$ with the state space $\mathcal{S}_1^*$ is given as follows:

$$\boldsymbol{P}^{(1)} = \begin{bmatrix} B_{00}^{(1)} & B_{01}^{(1)} & & & & \\ B_{10}^{(1)} & A_1^{(1)} & A_2^{(1)} & & & \\ & A_0^{(1)} & A_1^{(1)} & A_2^{(1)} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & A_1^{(1)} & A_2^{(1)} \\ & & & & A_0^{(1)} & C^{(1)} \end{bmatrix}$$

where $B_{00}^{(1)} = C^{*(1)}, B_{01}^{(1)} = D^{*(1)} \otimes \boldsymbol{\alpha}$ and $B_{10}^{(1)} = C^{*(1)} \otimes \mathbf{s}^0$; $A_0^{(1)} = C^{*(1)} \otimes \mathbf{s}^0 \boldsymbol{\alpha},\ A_1^{(1)} = D^{*(1)} \otimes \mathbf{s}^0 \boldsymbol{\alpha} + C^{*(1)} \otimes S,\ A_2^{(1)} = D^{*(1)} \otimes S$, and $C^{(1)} = D^{*(1)} \otimes \mathbf{s}^0 \boldsymbol{\alpha} + (1)_{1\times 1} \otimes S$, here $\mathbf{s}^0 = [0, \cdots, 0, 1]^T$. Let $\boldsymbol{\pi}_{|1} = (\pi_{|1}(0), \pi_{|1}(1), \cdots, \pi_{|1}(F_1 - 1))$ be the steady state probability vector of the Markov chain $\mathcal{M}_1$, where $\pi_{|1}(0) = (\pi_{|1}(0, 1, 0, 1))$ and $\pi_{|1}(l) = (\pi_{|1}(l, 1, d, 1), \cdots, \pi_{|1}(l, 1, 1, 1))$ for $l = 1, 2, \cdots, F_1 - 1$. By direct calculation of the equations $\boldsymbol{\pi}_{|1} = \boldsymbol{\pi}_{|1} \boldsymbol{P}^{(1)}$, $\boldsymbol{\pi}_{|1} \mathbf{e} = 1$, we have the following result.

Theorem 3.1-1 For $l = 1, 2, \cdots, F_1 - 1$, $s = 1, 2, \cdots, d$, $\pi_{|1}(l, 1, s, 1)$ can be expressed in terms of $\pi_{|1}(0, 1, 0, 1)$ as follows:

$$\pi_{|1}(1, 1, s, 1) = \frac{\lambda}{\bar{\lambda}^s} \pi_{|1}(0, 1, 0, 1), \qquad \pi_{|1}(2, 2, s, 1) = \left(\frac{\lambda}{\bar{\lambda}^{s+d}} - \frac{s\lambda^2}{\bar{\lambda}^{s+1}} - \frac{\lambda}{\bar{\lambda}^s}\right) \pi_{|1}(0, 1, 0, 1)$$

$$\begin{aligned} \pi_{|1}(l, 1, s, 1) = & \left(\frac{\lambda}{\bar{\lambda}^{(l-1)d+s}} - \left(\frac{((l-2)d + s)\lambda^2}{\bar{\lambda}^{(l-2)d+s+1}} + \frac{\lambda}{\bar{\lambda}^{(l-2)d+s}}\right) + \left(\frac{(\sum_{i=1}^{(l-3)d+s} i)\lambda^3}{\bar{\lambda}^{(l-3)d+s+2}} + \frac{((l-3)d + s)\lambda^2}{\bar{\lambda}^{(l-3)d+s+1}}\right) + \cdots \right. \\ & \left. + (-1)^l \left(\frac{b_{l,s}\lambda^{l-1}}{\bar{\lambda}^{d+s+l-2}} + \frac{b_{l-1,s}\lambda^{l-2}}{\bar{\lambda}^{d+s+l-3}}\right) + (-1)^{l+1} \left(\frac{(\sum_{i=1}^s a_{l,i})\lambda^l}{\bar{\lambda}^{s+l-1}} + \frac{(\sum_{i=1}^s a_{l-1,i})\lambda^{l-1}}{\bar{\lambda}^{s+l-2}}\right) \right) \pi_{|1}(0, 1, 0, 1) \end{aligned}$$

where $a_{2,s} = 1,\ a_{l,s} = \sum_{i=1}^{s} a_{l-1,i}$ *and* $b_{l,s} = \sum_{i=1}^{d} a_{l-1,i} + \sum_{i=1}^{s} a_{2,i} \sum_{i=1}^{d} a_{l-2,i} + \sum_{i=1}^{s} a_{3,i} \sum_{i=1}^{d} a_{l-3,i} + \ldots + \sum_{i=1}^{s} a_{l-5,i} \sum_{i=1}^{d} a_{5,i} + \sum_{i=1}^{s} a_{l-4,i} (\sum_{i=1}^{d} a_{4,i} + \sum_{i=1}^{d+1} i) + \sum_{i=1}^{s-1} a_{l-4,i} \sum_{i=1}^{d+2} i + \sum_{i=1}^{s-2} a_{l-4,i} \sum_{i=1}^{d+3} i + \ldots + \sum_{i=1}^{d+s} i.$ *Moreover,* $\pi_{|1}(0, 1, 0, 1)$ *can be determined by the normalization condition*

$$\pi_{|1}(0, 1, 0, 1) + \pi_{|1}(1, 1, 1, 1) + \ldots + \pi_{|1}(1, 1, d, 1) + \cdots + \pi_{|1}(F_1 - 1, 1, 1, 1) + \cdots + \pi_{|1}(F_1 - 1, 1, d, 1) = 1.$$

Next, we consider the cases $k = 2, 3, \cdots, K - 1$. For each k, $\mathcal{M}_k^*$ is a finite quasi-birth-death (QBD) process on the state space $\mathcal{S}_k^* = \{(l, j, s, k) \mid F_{k-1} \le l < F_k;\ 1 \le j \le k;\ 1 \le s \le d\}$ with transition probability matrix of the block tridiagonal form:

$$\boldsymbol{P}^{(k)} = \begin{bmatrix} B^{(k)} & A_2^{(k)} & & & \\ A_0^{(k)} & A_1^{(k)} & A_2^{(k)} & & \\ & A_0^{(k)} & \ddots & \ddots & \\ & & \ddots & A_1^{(k)} & A_2^{(k)} \\ & & & A_0^{(k)} & C^{(k)} \end{bmatrix}$$

where $B^{(k)} = (C^{*(k)} + D^{*(k)}) \otimes \mathbf{s}^0\boldsymbol{\alpha} + C^{*(k)} \otimes S$, $A_0^{(k)} = C^{*(k)} \otimes \mathbf{s}^0\boldsymbol{\alpha}$, $A_1^{(k)} = D^{*(k)} \otimes \mathbf{s}^0\boldsymbol{\alpha} + C^{*(k)} \otimes S$, $A_2^{(k)} = D^{*(k)} \otimes S$ and $C^{(k)} = D^{*(k)} \otimes \mathbf{s}^0\boldsymbol{\alpha} + (C^{*(k)} + D^{*(k)}) \otimes S$, all are $kd \times kd$ matrices. Let $\boldsymbol{\pi}_{|k} = (\boldsymbol{\pi}_{|k}(F_{k-1}), \boldsymbol{\pi}_{|k}(F_{k-1}+1), \cdots, \boldsymbol{\pi}_{|k}(F_k - 1))$ be the steady state probability vector of the Markov chain $\mathcal{M}_k^*$, where $\boldsymbol{\pi}_{|k}(l) = (\boldsymbol{\pi}_{|k}(l,d,k), \boldsymbol{\pi}_{|k}(l,d-1,k), \cdots, \boldsymbol{\pi}_{|k}(l,1,k))$ for $l = F_{k-1}, \cdots, F_k - 1$, and $\boldsymbol{\pi}_{|k}(l,s,k) = (\pi_{|k}(l,1,s,k), \pi_{|k}(l,2,s,k), \cdots, \pi_{|k}(l,k,s,k))$ for $1 \le s \le d$. Furthermore, let matrices $R_1^{(k)}$ and $R_2^{(k)}$ are, respectively, the unique minimal non-negative solutions of the quadratic matrix equations $R = A_2^{(k)} + RA_1^{(k)} + R^2A_0^{(k)}$ and $R = A_0^{(k)} + RA_1^{(k)} + R^2A_2^{(k)}$, and define the matrix $B[R_1^{(k)}, R_2^{(k)}]$ as follows:

$$B[R_1^{(k)}, R_2^{(k)}] = \begin{bmatrix} B^{(k)} + R_1^{(k)}A_0^{(k)} & (R_1^{(k)})^{F_k - F_{k-1} - 2}(A_0^{(k)} + R_1^{(k)}C^{(k)} - R_1^{(k)}) \\ (R_2^{(k)})^{F_k - F_{k-1} - 2}(R_2^{(k)}B^{(k)} + A_0^{(k)} - R_2^{(k)}) & C^{(k)} + R_2^{(k)}A_2^{(k)} \end{bmatrix}.$$

Applying Theorem 1 in [1] yields the following result.

Theorem 3.1-2 Let $\boldsymbol{v}^{(k)} = [\boldsymbol{v}_1^{(k)}, \boldsymbol{v}_2^{(k)}]$, where $\boldsymbol{v}_i^{(k)}, i = 1,2$ are a row vector of length d, be the left invariant eigenvector of $B[R_1^{(k)}, R_2^{(k)}]$, i.e., $\boldsymbol{v}^{(k)} = \boldsymbol{v}^{(k)}B[R_1^{(k)}, R_2^{(k)}]$ normalized so that $(\boldsymbol{v}_1^{(k)}T_1^{(k)} + \boldsymbol{v}_2^{(k)}T_2^{(k)})e_1 = 1$ where $T_i^{(k)} = \sum_{m=0}^{F_k - F_{k-1} - 1}(R_i^{(k)})^m$, $i = 1,2$. Then, the steady state probability vector $\boldsymbol{\pi}_{|k}$ are given by

$$\boldsymbol{\pi}_{|k}(F_{k-1} + l) = \boldsymbol{v}_1^{(k)}(R_1^{(k)})^l + \boldsymbol{v}_2^{(k)}(R_2^{(k)})^{F_k - F_{k-1} - (l+1)}, \qquad l = 0, 1, \cdots, F_k - F_{k-1} - 1.$$

Finally, the Markov chain $\mathcal{M}_K^*$ with the state space $\mathcal{S}_K^* = \{(l,j,s,K);\ F_{K-1} \le l < \infty;\ 1 \le j \le K,\ 1 \le s \le d\}$ is a infinite QBD process with the transition probability matrix of the form:

$$\boldsymbol{P}^{(K)} = \begin{pmatrix} B^{(K)} & A_2^{(K)} & & & \\ A_0^{(K)} & A_1^{(K)} & A_2^{(K)} & & \\ & A_0^{(K)} & A_1^{(K)} & A_2^{(K)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where $B^{(K)} = (C^{*(K)} + D^{*(K)}) \otimes \mathbf{s}^0\boldsymbol{\alpha} + C^{*(K)} \otimes S$, $A_0^{(K)} = C^{*(K)} \otimes \mathbf{s}^0\boldsymbol{\alpha}$, $A_1^{(K)} = D^{*(K)} \otimes \mathbf{s}^0\boldsymbol{\alpha} + C^{*(K)} \otimes S$ and $A_2^{(K)} = D^{*(K)} \otimes S$. Let $\boldsymbol{\pi}_{|K} = (\boldsymbol{\pi}_{|K}(F_{K-1}), \boldsymbol{\pi}_{|K}(F_{K-1}+1), \boldsymbol{\pi}_{|k}(F_{K-1}+2), \cdots\cdots)$ be the steady state probability vector of the Markov chain $\mathcal{M}_K^*$, where $\boldsymbol{\pi}_{|K}(l) = (\boldsymbol{\pi}_{|K}(l,d,K), \boldsymbol{\pi}_{|K}(l,d-1,K), \cdots, \boldsymbol{\pi}_{|K}(l,1,K))$ for $l = F_{K-1}, F_{K-1}+1, \cdots$, and $\boldsymbol{\pi}_{|K}(l,s,K) = (\pi_{|K}(l,1,s,K), \pi_{|K}(l,2,s,K), \cdots, \pi_{|K}(l,K,s,K))$ for $1 \le s \le d$. Furthermore, let matrix $R^{(K)}$ is the unique minimal non-negative solution of the following quadratic matrix equation $R = A_2^{(K)} + RA_1^{(K)} + R^2A_0^{(K)}$ and define the matrix $B[R^{(K)}] = B^{(K)} + R^{(K)}A_0^{(K)}$. By simple application of the results in Neuts [14], we have the following theorem.

Theorem 3.1-3 Let $\boldsymbol{\pi}_{|K}(F_{K-1})$ be the solution vector of the equations $\boldsymbol{\pi}_{|K}(F_{K-1}) = \boldsymbol{\pi}_{|K}(F_{K-1})B[R^{(K)}]$, $\boldsymbol{\pi}_{|K}(F_{K-1})(I - R^{(K)})^{-1}e_1 = 1$. Then, the steady state probability vectors $\boldsymbol{\pi}_{|K}(F_{K-1} + l)$ have the matrix geometric form

$$\boldsymbol{\pi}_{|K}(F_{K-1} + l) = \boldsymbol{\pi}_{|K}(F_{K-1})(R^{(K)})^l, \qquad l = 1, 2, \cdots.$$

3.3 The steady state probabilities of the aggregated Markov chain

Through Theorem 3.1-1 to 3.1-3, therefore, we have derived the steady state probability vectors $\boldsymbol{\pi}_{|k}$ for the Markov chains $\mathcal{M}_k^*$, $k = 1, 2, \cdots, K$, which are also the conditional steady state probabilities for the states in $\mathcal{S}_k^*$ of the associated Markov chain $\mathcal{M}^*$, given that the system is in partition $\mathcal{S}_k^*$. The remaining work is to find the steady state probability that the system is in $\mathcal{S}_k^*$. To do so, for $k = 1, 2, \cdots, K$, let us aggregate all the states in $\mathcal{S}_k^*$ into a single state (with a little abuse of notations, we denote the aggregated state still by $\mathcal{S}_k^*$) and define the steady state probabilities of the queue length as follows: $\pi_{|1}(l) = \sum_{s=1}^d \pi_{|1}(l,1,s,1)$ for $l = 0, 1, 2, \cdots, F_1 - 1$, $\pi_{|k}(F_{k-1} + l) = \sum_{j=1}^k \sum_{s=1}^d \pi_{|k}(F_{k-1} + l, j, s, k)$ for $l = 1, 2, \cdots, F_k - F_{k-1}$, $k = 2, 3, \cdots, K-1$ and $\pi_{|K}(F_{K-1} + l) = \sum_{j=1}^K \sum_{s=1}^d \pi_{|K}(F_{K-1} + l, j, s, K)$ for $l = 1, 2, \cdots$. Then, the resulting aggregated process is the Markov chain with the state space $\{\mathcal{S}_1^*, \mathcal{S}_2^*, \cdots, \mathcal{S}_K^*\}$ and transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 1 - \lambda\pi_{|1}(F_1 - 1) & \lambda\pi_{|1}(F_1 - 1) & & & \\ \bar{\lambda}\pi_{|2}(F_1) & 1 - \bar{\lambda}\pi_{|2}(F_1) - \lambda\pi_{|2}(F_2 - 1) & \lambda\pi_{|2}(F_2 - 1) & & \\ & \bar{\lambda}\pi_{|3}(F_2) & \cdots & & \\ & & \ddots & \ddots & \\ & & & \bar{\lambda}\pi_{|K}(F_{K-1}) & 1 - \bar{\lambda}\pi_{|K}(F_{K-1}) \end{pmatrix}.$$

The steady state distribution vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \cdots, \pi_K)$ of the aggregated Markov chain can be obtained by directly solving the equations $\boldsymbol{\pi} = \boldsymbol{\pi P}$, $\boldsymbol{\pi} e_K = 1$. We have the following theorem.

Theorem 3.2 The steady state probabilities π_k for $k = 1, 2, \cdots, K$ are determined by

$$\pi_1 = \left[1 + \sum_{k=2}^{K-1} (\frac{\lambda}{\bar{\lambda}})^{k-1} \prod_{l=1}^{k-1} \frac{\pi_{|l}(F_i - 1)}{\pi_{|l+1}(F_i)}\right]^{-1}, \qquad \pi_k = \left[1 + \sum_{k=2}^{K-1} (\frac{\lambda}{\bar{\lambda}})^{k-1} \prod_{l=1}^{k-1} \frac{\pi_{|l}(F_i - 1)}{\pi_{|l+1}(F_i)}\right]^{-1} (\frac{\lambda}{\bar{\lambda}})^{k-1} \prod_{l=1}^{k-1} \frac{\pi_{|l}(F_i - 1)}{\pi_{|l+1}(F_i)}.$$

Finally, the steady state probabilities of $\mathcal{M}^*$ can be expressed as

$$\pi(l, j, s, k) = \pi_{|k}(l, j, s, k)\pi_k, \qquad (l, j, s, k) \in \mathcal{S}^*.$$

4. Stationary waiting time distribution

In this section, we derive the stationary waiting time distribution of an arbitrary customer for the discrete-time $Geo/D/K$ queueing model with the multi-threshold service policy. As stated previously, this stationary waiting time distribution is the same as that of the last arriving customer in a group in the $D-MAP/D/1$ queueing system, whose arriving phase and bulk size (the number of active servers) are controlled by the thresholds F. Due to the variation of the number of active servers, the waiting time distribution of an arbitrary customer depends not only on the number of the customers ahead of him at arrival, but also on the number of the further arriving customers behind of him. Here we use the absorbing Markov chain method to compute the waiting time distribution. Take an arbitrary customer as the tagged one. Then, we introduce an absorbing discrete-time Markov chain to describe the dynamics of the system since the arrival instant of the tagged customer. As will be seen, the waiting time of this customer coincides with the time until the introduced Markov process reaching an absorption state. Thus, the resulting stationary waiting time distribution is of phase-type.

Let $\mathcal{Z} = \{(M_n, L_n, J_n, S_n, K_n),\ n \geq 0\}$, where M_n and L_n denote, respectively, the number of customers behind of the tagged customer and the number of customers ahead of the tagged customer; J_n, S_n and K_n denote, respectively, the phase of the Markovian arrival process, remaining group service time and the number of active servers at the nth slot. Note that 1) M_n as well as L_n do not include the tagged customer; 2) $M_0 = 0$ and 3) K_n represents the number of active servers immediately after a transfer of servers (i.e., addition or remove) if any. Especially, $K_0 = k - 1$ if $F_{k-1} \leq L_0 < F_k - 1$, and k if $L_0 = F_k - 1$. Then, $\mathcal{Z}$ is an Markov chain with the state space $\mathcal{T} = \bigcup_{k=1}^{K} (\mathcal{T}_k \cup \mathcal{T}_{ko})$ where $\mathcal{T}_k = \{(m, l, j, s, k) \mid m \geq 0,\ l \geq 1,\ F_{k-1} \leq m + l + 1 < F_k;\ \ 1 \leq j \leq k,\ 1 \leq s \leq d\}$ and $\mathcal{T}_{ko} = \{(m, 0, j, 0, k) \mid m \geq 0,\ F_{k-1} \leq m + 1 < F_k;\ \ 1 \leq j \leq k\}$. In fact, $\mathcal{T}_{ko}$ represents the set of the absorbing states when the number of active servers is k. Arranging the state sets in the order: $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_K; \mathcal{T}_{1o}; \mathcal{T}_{2o}, \cdots, \mathcal{T}_{Ko}$, we obtain the following transition probability matrix of $\mathcal{Z}$

$$\boldsymbol{Q} = \left[\begin{array}{cccc|cccc}
\boldsymbol{Q_{11}} & \boldsymbol{Q_{12}} & & & \boldsymbol{\mathcal{V}_{1o}} & & & \\
\boldsymbol{Q_{21}} & \boldsymbol{Q_{22}} & \boldsymbol{Q_{23}} & & & \boldsymbol{\mathcal{V}_{2o}} & & \\
 & \ddots & \ddots & & & & \ddots & \\
 & & \boldsymbol{Q_{KK-1}} & \boldsymbol{Q_{KK}} & & & & \boldsymbol{\mathcal{V}_{Ko}} \\
\hline
 & & & & \boldsymbol{I}_{F_1-1} & & & \\
 & & & & & \boldsymbol{I}_{2(F_2-F_1+1)} & & \\
 & & & & & & \ddots & \\
 & & & & & & & \boldsymbol{I}_{K(F_K-F_{K-1}+1)}
\end{array}\right]$$

where the sub-matrices $\boldsymbol{Q_{kk}}, \boldsymbol{Q_{kk+1}}, \boldsymbol{Q_{kk-1}}$ and $\boldsymbol{\mathcal{V}_{ko}}$ are given in Appendix.

Next, we estimate the initial distribution of the Markov chain $\mathcal{Z} = \{(M_n, L_n, J_n, S_n, K_n),\ n \geq 0\}$. Let $x_{(0,0,1,0,1)} = P((M_0, L_0, J_0, S_0, K_0) = (0, 0, 1, 0, 1))$ and $x_{(m,l,j,s,k)} = P((M_0, L_0, J_0, S_0, K_0) = (m, l, j, s, k))$ for $0 \leq m;\ 0 \leq l;\ 1 \leq j \leq k;\ 1 \leq s \leq d;\ 1 \leq k \leq K$. Furthermore, define the vectors $\boldsymbol{x}_{(m,l,j,k)} = (x_{(m,l,j,d,k)}, \cdots, x_{(m,l,j,1,k)})$ and $\boldsymbol{\pi}(l, j, k) = (\pi(l, j, d, k), \cdots, \pi(l, j, 1, k))$. Note that $M_0 = 0$ when the tagged customer arrives at the system. We have $\boldsymbol{x}_{(m,l,j,k)} = \boldsymbol{0}$

for all $m > 0$. Hence, we only need to consider the case $m = 0$. These initial probability vectors can be obtained from

$$x_{(0,0,1,0,1)} = \frac{1}{\lambda}\left(\pi(0,1,0,1)D^{*(1)} + \boldsymbol{\pi}(1,1,1)D^{*(1)} \otimes \mathbf{s}^0\right) = \pi(0,1,0,1) + \pi(1,1,d,1)$$

$$\boldsymbol{x}_{(0,l,j,k)} = \frac{1}{\lambda}\left(\boldsymbol{\pi}(l,j,k)D^{*(k)} \otimes S + \boldsymbol{\pi}(l+1,j,k)D^{*(k)} \otimes \mathbf{s}^0\boldsymbol{\alpha}\right) \qquad F_{k-1} \le l < F_k - 1;\ 1 \le k \le K-1$$

$$\boldsymbol{x}_{(0,l,j,K)} = \frac{1}{\lambda}\left(\boldsymbol{\pi}(l,j,K)D^{*(k)} \otimes S + \boldsymbol{\pi}(l+1,j,K)D^{*(k)} \otimes \mathbf{s}^0\boldsymbol{\alpha}\right), \qquad F_{K-1} \le l < \infty.$$

Finally, we can identify the stationary waiting time distribution of the tagged customer with the phase distribution $(\boldsymbol{x}, Q)$. The precise algorithm for computing this stationary distribution can also be established.

[*Algorithm for the stationary waiting time distribution*]

STEP.1 Calculate the steady state probability vector $\boldsymbol{\pi}_{|1}$ according to Theorem 3.1-1.

STEP.2 (1) Calculate matrix geometric factors $R_1^{(k)}$ and $R_2^{(k)}$ for $k = 2, \cdots, K$

(i) calculate the matrices: $H_0^{(k)} = A_0^{(k)} + A_1^{(k)} + A_2^{(k)} - I^{(k)}$, $H_1^{(k)} = 2(A_0^{(k)} - A_2^{(k)})$ and $H_2^{(k)} = A_2^{(k)} - A_1^{(k)} + A_0^{(k)} + I^{(k)}$, $G_i^{(k)} = H_i^{(k)}(H_2^{(k)})^{-1}$, $i = 0, 1$, where $I^{(k)}$ denotes the $kd \times kd$ identity matrix. Define the $2kd \times 2kd$ matrix $E^{(k)}$ such as $E^{(k)} = \begin{bmatrix} \mathbf{0} & -G_0^{(k)} \\ I^{(k)} & -G_1^{(k)} \end{bmatrix}$.

(ii) Let $\boldsymbol{y}^{(k)}$ be an invariant vector satisfying the equations: $\boldsymbol{y}^{(k)}(A_0^{(k)} + A_1^{(k)} + A_2^{(k)}) = \boldsymbol{y}^{(k)}$, $\boldsymbol{y}^{(k)}e_k = 1$, where e_k is a column vector of ones of length kd. Then, we have $\boldsymbol{y}^{(k)} = (1/kd, 1/kd, \cdots, 1/kd)$. Define the row vector $\boldsymbol{\alpha}^{(k)} = (\boldsymbol{y}^{(k)}, \mathbf{0})_{2kd\times 1}$ and the column vector $\boldsymbol{\beta}^{(k)} = \begin{bmatrix} H_1^{(k)}e_k \\ H_2^{(k)}e_k \end{bmatrix}_{2kd\times 1}$. Define the $2kd \times 2kd$ matrix $E_m^{(k)} = E^{(k)} + \dfrac{\boldsymbol{\beta}^{(k)}\boldsymbol{\alpha}^{(k)}}{\boldsymbol{\alpha}^{(k)}\boldsymbol{\beta}^{(k)}}$.

(iii) Find the bases of the matrix $(E_m^{(k)})^\tau$ by the matrix sign function approach. Denote the left bases and right bases, respectively, by $V^{(k)} = \begin{bmatrix} V_1^{(k)} \\ V_2^{(k)} \end{bmatrix}$ and $U^{(k)} = \begin{bmatrix} U_1^{(k)} \\ U_2^{(k)} \end{bmatrix}$.

(iv) Calculate $R_1^{(k)} = (V_1^{(k)} - V_2^{(k)})^{-\tau}(V_1^{(k)} + V_2^{(k)})^\tau$ and $R_2^{(k)} = (U_1^{(k)} - U_2^{(k)})^{-\tau}(U_1^{(k)} + U_2^{(k)})^\tau$.

(2) Calculate the steady state probability vector $\boldsymbol{\pi}_{|k}$ for $k = 2, \cdots, K-1$ and $\boldsymbol{\pi}_{|K}$, respectively according to Theorem 3.1-2 and 3.1-3.

STEP.3 Calculate the steady state probability vector $\{\pi_1, \pi_2, \cdots, \pi_K\}$ and $\pi(l, j, s, k)$ according to Theorem 3.2.

STEP.4 Calculate the initial distribution $x_{(0,0,1,0,1)}$, $\boldsymbol{x}_{(0,l,j,k)}$ and $\boldsymbol{x}_{(0,l,j,K)}$.

STEP.5 Finally, calculate the stationary waiting time distribution from $(\boldsymbol{x}, Q)$.

Some numerical examples have been performed. We omit to show the results here due to the page limitation.

5. Conclusion

In this paper we have analyzed the discrete-time $Geo/D/K$ queue with the multi-threshold service policy. We derived the steady state distribution of the queue length for the corresponding discrete-time $MAP/D/1$ queueing system using Crommelin's and Stochastic complement techniques, and utilizing this steady-state distribution and the absorbing Markov process method we identify the stationary waiting time distribution with the phase type. Furthermore, we presented the algorithm for computing the steady-state distribution. As known, the discrete-time Markovian arrival process (D-MAP) has been extensively introduced because of the limitation of Bernoulli processes in modeling the correlated arrivals. As the further work, we will consider the analysis for the waiting time distribution of the $D-MAP/D/K$ queuing system under the multi-threshold service policy.

Acknowledge This work was supported in part by Grant-in-Aid for Scientific Research(No.21510143), Ministry of Education, Culture, Sports, Science and Technology, Japan.

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Appendix [The definition of the matrices $\boldsymbol{Q}_{kk}, \boldsymbol{Q}_{kk+1}, \boldsymbol{Q}_{kk-1}$ and $\boldsymbol{\mathcal{V}}_{ko}$]

$$\boldsymbol{Q}_{kk} = \begin{bmatrix} \boldsymbol{Q}_{kk}^{(0,0)} & \boldsymbol{Q}_{kk}^{(0,1)} & & & \\ & \boldsymbol{Q}_{kk}^{(1,1)} & \boldsymbol{Q}_{kk}^{(1,2)} & & \\ & & \ddots & \ddots & \\ & & & & \boldsymbol{Q}_{kk}^{(F_k-3,F_k-3)} \end{bmatrix}, \quad \boldsymbol{Q}_{kk+1} = \begin{bmatrix} \mathbf{0} & \boldsymbol{Q}_{kk+1}^{(0)} & & & \mathbf{0} & \cdots & \mathbf{0} \\ & \mathbf{0} & \boldsymbol{Q}_{kk+1}^{(1)} & & \mathbf{0} & \cdots & \mathbf{0} \\ & & \ddots & \ddots & \vdots & & \vdots \\ & & & \mathbf{0} \quad \boldsymbol{Q}_{kk+1}^{(F_k-3)} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{Q}_{kk-1} = \begin{bmatrix} \boldsymbol{Q}_{kk-1}^{(0)} & & & \\ & \boldsymbol{Q}_{kk-1}^{(1)} & & \\ & & \ddots & \\ & & & \boldsymbol{Q}_{kk-1}^{(F_{k-1}-3)} \\ \mathbf{0}_{(F_k-F_{k-1})kd\times(F_{k-1}-2)kd} & \cdots & \cdots & \mathbf{0}_{(F_k-F_{k-1})kd\times kd} \\ \vdots & & & \vdots \\ \mathbf{0}_{kd\times(F_{k-1}-2)kd} & \cdots & \cdots & \mathbf{0}_{kd\times kd} \end{bmatrix}$$

here for $i = 0, 1, \cdots, F_k - 3$,

$$\boldsymbol{Q}_{kk}^{(i,i)} = C^{*(k)} \otimes \mathbf{S} \otimes \mathbf{I}_k^{(i)} + C^{*(k)} \otimes \mathbf{s}^0 \boldsymbol{\alpha} \otimes \begin{bmatrix} \mathbf{0} & \mathbf{J}_k^{(i)} \\ 0 & \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{Q}_{kk}^{(i,i+1)} = D^{*(k)} \otimes \mathbf{s}^0 \boldsymbol{\alpha} \otimes \begin{bmatrix} \mathbf{J}_k^{(i)} \\ \mathbf{0} \end{bmatrix} + D^{*(k)} \otimes \mathbf{S} \otimes \begin{bmatrix} \mathbf{0} \\ \mathbf{J}_k^{(i)} \end{bmatrix}$$

$$\mathbf{I}_k^{(i)} = \begin{cases} \mathbf{I}_{F_k-F_{k-1}} & \text{for } i = 0, 1, \cdots, F_{k-1} - 2 \\ \mathbf{I}_{F_k-(i+2)} & \text{for } i = F_{k-1} - 1, \cdots, F_k - 3 \end{cases}, \qquad \mathbf{J}_k^{(i)} = \begin{cases} \mathbf{I}_{F_k-F_{k-1}-1} & \text{for } i = 0, 1, \cdots, F_{k-1} - 2 \\ \mathbf{I}_{F_k-(i+3)} & \text{for } i = F_{k-1} - 1, \cdots, F_k - 3. \end{cases}$$

and

$$\boldsymbol{Q}_{kk+1}^{(i)} = D^{*(k)} \otimes \mathbf{S} \otimes \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{(F_k-(i+2))\times(F_{k+1}-F_k-1)}, \qquad \boldsymbol{Q}_{kk-1}^{(i)} = C^{*(k)} \otimes \mathbf{s}^0 \boldsymbol{\alpha} \otimes \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}_{(F_k-F_{k-1})\times(F_{k-1}-(i+2))}.$$

$\mathcal{V}_{ko} = \left[\mathcal{V}_{ko}^{(0)} \cdots \mathcal{V}_{ko}^{(F_{k-1}-3)} \cdots \mathcal{V}_{ko}^{(F_k-3)} \right]^{\tau}$, where τ denotes the matrix transpose and for $i = 0, 1, \cdots, F_{k-1} - 3$, $\mathcal{V}_{ko}^{(i)} = \mathbf{0}$, and for $i = F_{k-1} - 2, \cdots, F_k - 3$

$$\mathcal{V}_{Ko}^{(i)} = \begin{bmatrix} \mathcal{V}_{Ko}^{(i)}(F_k-(i+2), F_{k-1}-2) & \mathcal{V}_{Ko}^{(i)}(1, F_{k-1}-1) & \cdots & \mathcal{V}_{Ko}^{(i)}(F_k-(i+2), F_k-2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{V}_{Ko}^{(i)}(2, F_{k-1}-2) & \mathcal{V}_{Ko}^{(i)}(2, F_{k-1}-1) & \cdots & \mathcal{V}_{Ko}^{(i)}(2, F_k-2) \\ \mathcal{V}_{Ko}^{(i)}(1, F_{k-1}-2) & \mathcal{V}_{Ko}^{(i)}(1, F_{k-1}-1) & \cdots & \mathcal{V}_{Ko}^{(i)}(1, F_k-2) \end{bmatrix}$$

here for $m = 1, 2, \cdots, F_k - (i+2)$; $j = F_{k-1} - 2, F_{k-1} - 1, \cdots, F_k - 2$

$$\mathcal{V}_{ko}^{(i)}(m, j) = \begin{cases} C^{*(k)} & \text{if } (m, j) = (1, i) \\ D^{*(k)} & \text{if } (m, j) = (1, i+1) \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad \text{Furthermore,} \qquad \boldsymbol{Q}_{KK} = \begin{bmatrix} \boldsymbol{Q}_{KK}^{(0,0)} & \boldsymbol{Q}_{KK}^{(0,1)} & & \\ & \boldsymbol{Q}_{KK}^{(1,1)} & \boldsymbol{Q}_{KK}^{(1,2)} & \\ & & \ddots & \ddots \end{bmatrix}$$

where for $i = 0, 1, 2, \cdots$,

$$\boldsymbol{Q}_{KK}^{(i,i)} = C^{*(K)} \otimes \mathbf{S} \otimes \mathbf{I}_K^{(i)} + C^{*(K)} \otimes \mathbf{s}^0 \boldsymbol{\alpha} \otimes \begin{bmatrix} \mathbf{0} & \mathbf{J}_K^{(i)} \\ 0 & \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{Q}_{KK}^{(i,i+1)} = D^{*(K)} \otimes \mathbf{s}^0 \boldsymbol{\alpha} \otimes \begin{bmatrix} \mathbf{J}_K^{(i)} \\ \mathbf{0} \end{bmatrix} + D^{*(K)} \otimes \mathbf{S} \otimes \begin{bmatrix} \mathbf{0} \\ \mathbf{J}_K^{(i)} \end{bmatrix}$$

and $\mathbf{I}_K^{(i)}$ and $\mathbf{J}_K^{(i)}$ are both the infinite dimensional identify matrices such that

$$\mathbf{I}_K^{(i)} = \begin{bmatrix} \ddots & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \qquad \mathbf{J}_K^{(i)} = \begin{bmatrix} \ddots & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

The superscript i denotes that the dimensional numbers of $\mathbf{I}_K^{(i)}$ and $\mathbf{J}_K^{(i)}$ are counted from $F_{K-1} - (i+1)$ and from $F_{K-1} - (i+2)$ to infinite, respectively.

$$\boldsymbol{Q}_{KK-1} = \begin{bmatrix} \boldsymbol{Q}_{KK-1}^{(0)} & & & \\ & \boldsymbol{Q}_{KK-1}^{(1)} & & \\ & & \ddots & \\ & & & \boldsymbol{Q}_{KK-1}^{(F_{K-1}-3)} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & & & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} \end{bmatrix}$$

where for $i = 0, 1, \cdots, F_{K-1} - 3$

$$\boldsymbol{Q}_{KK-1}^{(i)} = C^{*(K)} \otimes \mathbf{s}^0 \boldsymbol{\alpha} \otimes \begin{bmatrix} \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}_{(F_K - F_{K-1}) \times (F_{K-1} - (i+2)).} \qquad \mathcal{V}_{Ko} = \begin{bmatrix} \mathcal{V}_{Ko}^{(0)} \\ \mathcal{V}_{Ko}^{(1)} \\ \vdots \\ \mathcal{V}_{Ko}^{(F_{K-1}-1)} \\ \vdots \end{bmatrix}$$

where for $i = 0, 1, \cdots, F_{k-1} - 3$, $\mathcal{V}_{Ko}^{(i)} = \mathbf{0}$, and for $i = F_{K-1} - 2, F_{K-1} - 1, \cdots$ and $m = 1, 2, \cdots$; $j = i, i+1, \cdots$

$$\mathcal{V}_{Ko}^{(i)} = \begin{bmatrix} \vdots & \vdots & \cdots \\ \mathcal{V}_{Ko}^{(i)}(2, i) & \mathcal{V}_{Ko}^{(i)}(2, i+1) & \cdots \\ \mathcal{V}_{Ko}^{(i)}(1, i) & \mathcal{V}_{Ko}^{(i)}(1, i+1) & \cdots \end{bmatrix}, \qquad \mathcal{V}_{Ko}^{(i)}(m, j) = \begin{cases} C^{*(K)} & \text{if } (m, j) = (1, i) \\ D^{*(K)} & \text{if } (m, j) = (1, i+1) \\ \mathbf{0} & \text{otherwise} \end{cases}.$$